*Journal of Applied Mechanics and Technical Physics, Vol. 39, No. 1, 1998*

# RHEOLOGICAL MODEL OF VOLUME STRETCHING OF NEWTONIAN LIQUIDS

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UDC 532.135:532.528

*An equation of state for a volumetrically stretched cavitating liquid medium that holds in the entire span of volume concentrations of bubbles ranging from cavitation nuclei to the stage of formation of a cellular foam structure is obtained based on a proposed macrorheological model. The dependence of the modulus of volume elasticity of a liquid on the volume concentration of bubbles is plotted, and a method for estimating the relaxation time for tensile stresses in cavitating liquid media is proposed.*

Since volume stretching of liquid media is always accompanied by development of cavitation, which leads to fragmentation of the medium in the case of unrestricted growth of cavitation bubbles, the need arises to create physical and mathematical models of such processes.

It is known that, within the framework of the three-dimensional theory of linear viscoelasticity, the viscoelastic behavior of a medium is usually considered under conditions of pure tension and shear, and the results obtained enable one to develop a general theory. (In this approach, stress and strain tensors are grouped into isotropic tensors and deviators, and viscoelasticity relations are written for each case.) It is expedient to use this approach in developing a rheological model of cavitating liquid media as well. The author [1] derived a relaxation-type rheological equation of state that characterizes the behavior of liquid media containing disperse elements and cavitation bubbles, in the regime of shear strains for any fixed value of the volume concentration of the bubbles $\alpha_0$. To create a mathematical model of volumetrically stretched liquid media, it is necessary to derive a rheological equation of state for the medium that depends on the varying (increasing in the process of volume stretching of the medium) concentration of cavitation bubbles in the entire range of its occurrence. Below, we consider a mechanical model for volumetrically stretched Newtonian liquids and derive a rheological equation of state for these liquids.

**1.** As has been shown [2–4], in the process of volume stretching, a liquid medium traverses a number of rheological states because of the unrestricted growth of cavitation bubbles from an almost ideal fluid to the state of a viscoelastic medium. We consider the initial stage of stretching ($\alpha_{00} = 10^{-12}$–$10^{-4} < \alpha_0 < 10^{-1}$) in which growth of the effective shear viscosity of the medium, which is caused by the presence of bubbles, is not yet taken into account, and the bubbles are assumed not to interact with one another and to exert an effect on each other only via the averaged field of pressures in the liquid. At this stage of the process, one can use the Iordanskii–Kogarko mathematical model [5, 6], which was constructed for an ideal incompressible fluid uniformly filled with spherical bubbles. The medium's compressibility is due to the bubbles' compressibility, the dynamics of which is described by the Lamb–Rayleigh equation.

If a liquid matrix did not contain cavitation nuclei ($\alpha_{00} = 0$), it would store elastic energy during volume stretching, as shown by the mechanical model in Fig. 1a. The time of partial relaxation of tensile stresses owing to restructuring of this medium at the molecular level, i.e., without a change in volume, can be estimated by the formula [7]

$$T_0 = \zeta_0/(K_\infty - K_0), \tag{1.1}$$

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 39, No. 1, pp. 35–46, January–February, 1998. Original article submitted June 26, 1996.

 0021-8944/98/3901-0030 \$20.00 

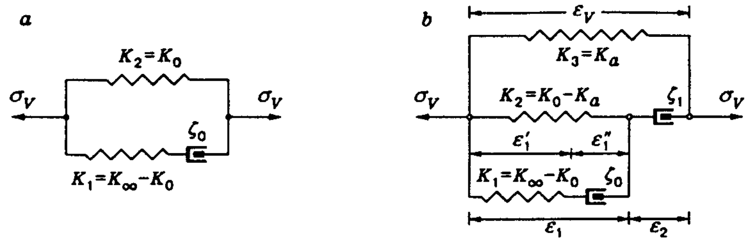


Fig. 1

where $\zeta_0$ is the volume (second) viscosity of the liquid, and $K_0$ and $K_\infty$ are, respectively, the static and dynamic bulk moduli of the liquid without development of cavitation bubbles. This process can occur for $\mathrm{De} = T_0/\Delta t_* \gg 1$, where De is the Deborah number and $\Delta t_*$ is the characteristic time of stretching of the medium [according to (1.1), $T_0 \cong 10^{-10}$ sec in the case of water]. If pulsed stretching of the medium occurs in the regime $\mathrm{De} < 1$, cavitation bubbles begin to grow in the process of deformation: the elastic energy of the medium is expended on development of the cavitation process.

The author [8] considered the problem of stretching a cylindrical water column containing cavitation nuclei along the axis of symmetry within the framework of the Iordanskii–Kogarko model. The process occurred owing to an instantaneously applied constant acceleration. A dependence was obtained that allows one to estimate the time of relaxation of tensile stresses $T$, which is due to growth of cavitation bubbles in the plane of loading, i.e., in a liquid layer. According to the estimate for $\alpha_{00} = 10^{-4}$, a radius of monodisperse cavitation nuclei $R_0 = 10^{-5}$ cm, and a tensile stress of $-30$ MPa, we have $T = 6.3 \cdot 10^{-9}$ sec.

Since $T$ is always much shorter than the value attainable in experiments $\Delta t_*$, Stanyukovich [9] and Chernobaev [10] used the assumption of instantaneous relaxation of tensile stresses. The essence of this assumption is that after a certain level of negative pressure is reached in the liquid, this pressure immediately relaxes and the cavitating liquid volume expands as a substance having no strength or viscosity. This model holds if the change in the rheological parameters (viscosity, elasticity, etc.) can be ignored in the process of the medium's deformation, as, for example, in problems of the dynamics of the cavitation zone for $\alpha_0 \leqslant 0.1$.

However, as was shown in [2–4], as $\alpha_0$ grows owing to an increase in the effective shear viscosity $\mu$ and a decrease in the dynamic shear modulus $G_\infty$, the relaxation time of shear stresses $\lambda_0 = \mu/G_\infty$ increases. For example, for $\alpha_0 \to \alpha_{0*} = 0.74$–$0.77$ (the concentration of the limiting bubble packing), we have $\lambda_0(\alpha_0)/\lambda_0(\alpha_{0*}) \to 10^4$. Later on, at the stage $\alpha_0 > \alpha_{0*}$, the bubble system begins to be structured, namely, a cellular foam structure is formed, and the medium loses the property of fluidity, entering the state of a viscoelastoplastic body [1].

**2.** With the aforesaid taken into account, the mechanical model of a volumetrically stretched cavitating medium can be shown schematically, as is done in Fig. 1b. Here the model of a pure liquid (Fig. 1a) is supplemented by the elements $K_3$ and $\zeta_1$, which correspond to the bulk modulus $K_a$ and the effective bulk viscosity of the cavitating liquid. The model works as follows. If the rate of increase of the tensile stress is rather high, the resistance to displacement of the pistons $\zeta_0$ and $\zeta_1$ is so great that at the initial moment these pistons can be regarded as nondeformable elements, and the deformation of the model is determined by the elasticity of the elements $K_1 = K_\infty - K_0$, $K_2 = K_0 - K_a$, and $K_3 = K_a$, i.e., at the initial moment the resulting elasticity of the model $K = K_1 + K_2 + K_3 = K_\infty$ corresponds to the dynamic bulk modulus of a pure liquid.

In Fig. 2, which is a qualitative illustration of the dependence of the tensile stresses $\sigma_V$ on the bulk tensile strains $\varepsilon_V$, this stage of the process corresponds to section 0–1, where the strain grows according to the law $\sigma_V = K_\infty \varepsilon_V$. Next, because, according to [11], for a bubble suspension we have $\zeta_1 > \zeta_0$, the spring $K_1$ begins to unload first in the model shown in Fig. 1b. The relaxation time of stresses in this stage is determined by formula (1.1). In Fig. 2, this process corresponds to section 1–2.

After that, the piston $\zeta_1$ begins to move, thus unloading the spring $K_2$, which corresponds to the loss

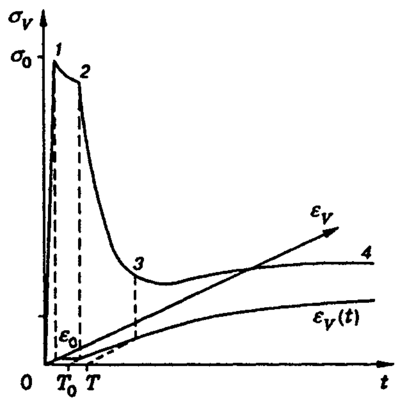


Fig. 2

of elastic energy of the liquid matrix because of the expansion of the nuclei of cavitation bubbles and the divergent motion of the layers of liquid with shear viscosity $\mu_0$ that are joined to these nuclei. This relaxation stage of elastic tensile stresses in the medium corresponds to section 2–3 in Fig. 2.

The subsequent volume stretching of the two-phase medium has an inertial character and occurs owing to growth of cavitation bubbles with retention of the volume of the liquid matrix, i.e., the springs $K_1$ and $K_2$ do not deform in the mechanical model but the element $K_3$ continues to stretch, and the piston $\zeta_1$ continues to move. At this stage, the medium still possesses some elasticity owing to the counter pressure of the ambient atmosphere, the gas and vapor pressure in the bubbles, and the surface tension, which is incorporated in the spring's elasticity $K_3 = K_a$ in the mechanical model (Fig. 1b). Section 3–4 in Fig. 2 corresponds to this stage. If $\alpha_0$ reaches the value $\alpha_{0*}$ and the bubbles come into contact with one another, and the medium enters the state of a cellular foam skeleton for $\alpha_0 > \alpha_{0*}$. Thus, we have $K_3 = K_a(\alpha_{0i})$, where $0 < \alpha_{0i} < 1$. We note that since the relaxation process in a pure liquid can end simultaneously with the onset of stress relaxation caused by expansion of cavitation bubbles, the total relaxation time of volumetric tensile stresses depends on all the viscous and elastic elements.

**3.** Based on the mechanical model (Fig. 1b), we construct a rheological equation of state for a volumetrically stretched liquid medium.

If $\sigma_V$ and $\varepsilon_V$ are the volumetric tensile stress and strain of the liquid medium, respectively, we have, according to the scheme in Fig. 1b,

$$\sigma_V = \sigma_1 + \sigma_2, \tag{3.1}$$

where

$$\begin{gathered} \sigma_1 = K_3\varepsilon_V = K_a(\varepsilon_1 + \varepsilon_2), \qquad \sigma_2 = \sigma_2' + \sigma_2'', \qquad \sigma_2' = K_2\varepsilon_1, \\ \varepsilon_1 = \varepsilon_1' + \varepsilon_1'', \quad \sigma_2'' = K_1\varepsilon_1', \quad \sigma_2'' = \zeta_0\partial\varepsilon_1''/\partial t. \end{gathered} \tag{3.2}$$

It follows from the above relation that

$$\varepsilon_1 = \left(\frac{1}{K_1} + \frac{1}{\zeta_0\partial/\partial t}\right)\sigma_2'' \quad \text{or} \quad \sigma_2'' = \varepsilon_1\left(\frac{1}{K_1} + \frac{1}{\zeta_0\partial/\partial t}\right)^{-1}.$$

Hence,

$$\sigma_2 = \sigma_2' + \sigma_2'' = \left[K_2 + \left(\frac{1}{K_1} + \frac{1}{\zeta_0\partial/\partial t}\right)^{-1}\right]\varepsilon_1,$$

and, expressing $\varepsilon_1$ with allowance for $\sigma_2 = \zeta_1\,\partial\varepsilon_2/\partial t$ and $\varepsilon_V = \varepsilon_1 + \varepsilon_2$, we obtain

$$\sigma_2 = \left\{\left[K_2 + \left(\frac{1}{K_1} + \frac{1}{\zeta_0\partial/\partial t}\right)^{-1}\right]^{-1} + \frac{1}{\zeta_1\partial/\partial t}\right\}^{-1}\varepsilon_V. \tag{3.3}$$

Having substituted (3.2) and (3.3) into (3.1), we have

$$\sigma_V = K_a \varepsilon_V + \left\{ \left[ K_2 + \left( \frac{1}{K_1} + \frac{1}{\zeta_0 \partial/\partial t} \right)^{-1} \right]^{-1} + \frac{1}{\zeta_1 \partial/\partial t} \right\}^{-1} \varepsilon_V.$$

From the above relation, with allowance for the fact that $K_1 = K_\infty - K_0$, $K_2 = K_0 - K_a$, and $T_0 = \zeta_0/(K_\infty - K_0)$ and denoting $T_1 = \zeta_1/(K_0 - K_a)$, we obtain, after transformation, the following rheological equation of state for a volumetrically stretched liquid medium, which holds for $\alpha_{00} \leqslant \alpha_{0i} < 1$:

$$\ddot{\sigma}_V + \left( \frac{1}{T_0} + \frac{K_\infty - K_a}{K_0 - K_a} \frac{1}{T_1} \right) \dot{\sigma}_V + \frac{\sigma_V}{T_0 T_1} = K_\infty \ddot{\varepsilon}_V + \left( \frac{K_0}{T_0} + \frac{K_\infty - K_a}{K_0 - K_a} \frac{K_a}{T_1} \right) \dot{\varepsilon}_V + \frac{K_a}{T_0 T_1} \varepsilon_V. \quad (3.4)$$

Since $K_a = K_a(\alpha_{0i})$ and $\zeta_1 = \zeta_1(\alpha_{0i})$, Eq. (3.4) can be solved numerically if the dependences of the rheological parameters of the medium on the growth of the volume concentration of bubbles are found.

**4.** We consider the bulk modulus of a bubbly suspension versus $\alpha_0$.

*The stage* $0 \leqslant \alpha_0 < \alpha_{0*}$. As is known, the bulk modulus of a pure liquid

$$K = -V dP/dV \quad (4.1)$$

characterizes its elasticity in the vicinity of a prescribed pressure. If the liquid contains a bubble, the balanced-state condition for the bubble can be written [7], under a given pressure in the liquid $P = P^0$, as follows:

$$P^0 = P_1^0 + P_2 - 2\gamma/r^0, \quad (4.2)$$

where $P_1^0$ and $P_2$ are the pressures of the gas and saturated vapor in the bubble, respectively, $r^0$ is the bubble radius, and $\gamma$ is the surface tension of the liquid. According to (4.2), if the pressure in the liquid decreases to $P' < P^0$, the bubble radius will increase to $r'$, so that $z = r'/r^0 > 1$. The gas pressure in the bubble will decrease according to the law $P_1' = P_1^0/z^3$, whereas $P_2$ will remain unchanged (in the expansion time of the bubble to the radius $r'$, diffusion of liquid vapor through the bubble wall "equalizes" $P_2$). Expressing $P_1^0 = P^0 - P_2 + 2\gamma/r^0$ from (4.2), we write

$$P' = P_1' + P_2 - \frac{2\gamma}{r'} = \frac{1}{z^3} \left( P^0 - P_2 + \frac{2\gamma}{r^0} \right) - \frac{2\gamma}{z r^0} + P_2.$$

It follows that

$$\left. \frac{dP'}{dz} \right|_{z=1} = -3 \left( P^0 - P_2 + \frac{2\gamma}{r^0} \right) + \frac{2\gamma}{r^0}. \quad (4.3)$$

Assuming that the liquid medium contains $N$ monodisperse bubbles of radius $r^0$, we can write an expression for the volume of the two-phase medium in the form

$$V^0 = V_0 + \frac{4}{3}\pi (r^0)^3 N, \quad (4.4)$$

where $V_0$ is the volume of the pure liquid component and $N$ is such that $\alpha_0 < \alpha_{0*}$. If the bubble radius increases to $r'$ owing to a pressure drop, relation (4.4) takes the form

$$V^{0\prime} = V_0 + V_1 = V_0 + (4/3)\pi r'^3 N.$$

It follows that

$$dV^0 = dV_0 + (3/z) V_1 \, dz. \quad (4.5)$$

With allowance for (4.3) and the relations $\alpha_0 = V_1/V^0$ and $1 - \alpha_0 = V_0/V^0$, we obtain from (4.1) and (4.5) an expression for the bulk modulus of a liquid containing bubbles:

$$K_a = -V^0 \frac{dP}{dV^0} = -V^0 \frac{dP}{dV_0 + (3/z) V_1 \, dz} \quad (4.6)$$

$$= -\left( \frac{1}{V^0} \frac{dV_0}{dP} + \frac{3}{z} \alpha_0 \frac{dz}{dP} \right)^{-1} \Bigg|_{z=1} = \left[ (1 - \alpha_0)\chi_0 + \frac{3\alpha_0}{3(P^0 - P_2) + 4\gamma/r^0} \right]^{-1}, \quad (4.6)$$

where $\chi_0 = -(1/V_0)\, dV_0/dP = 1/K_0$ is the compressibility coefficient of the pure liquid.

Since $\alpha_0$ and $r^0$ are interrelated, for definiteness we express this relationship via the number concentration of bubbles $n = N/V^0$, which is true in the absence of bubble coalescence in the process of stretching the medium. (This assumption is reasonable at least at the stage $\alpha_0 < \alpha_{0*}$ when the volume is stretched monotonically.) From the condition $n_i V_i^0 = N = \text{const}$, with allowance for (4.4) we have

$$n_i = n_j \frac{V_j^0}{V_i^0} = n_j \frac{V_0 + (4/3)\,\pi(r_j^0)^3 V_j^0 n_j}{V_0 + (4/3)\,\pi(r_i^0)^3 V_j^0 n_j} = n_j \frac{V_0/V_j^0 + \alpha_{0j}}{V_0/V_j^0 + \alpha_{0j}(r_i^0/r_j^0)^3},$$

or after $\alpha_{0j}$ and $n_j$ are replaced by the initial quantities $\alpha_{00}$ and $n_0$, respectively, with allowance for $V_j^0 = V^0/(1-\alpha_{0j})$, we have

$$n_i = \frac{n_0}{1 + [(r_i/r_0)^3 - 1]\alpha_{00}}, \tag{4.7}$$

where $r_0$ and $r_i$ are the initial and current radii of the bubbles.

Substituting the following expression, which was derived with allowance for (4.7), into (4.6):

$$\alpha_0 = \alpha_{0i} = \frac{4}{3}\pi r_i^3 \frac{n_0}{1 + [(r_i/r_0)^3 - 1]\alpha_{00}} = \frac{\alpha_{00}(r_i/r_0)^3}{1 + [(r_i/r_0)^3 - 1]\alpha_{00}}, \tag{4.8}$$

we write

$$K_a = 1\Big/\left\{\chi_0 + \left[\frac{3}{3(P^0 - P_2) + 4\gamma/r_i} - \chi_0\right]\frac{\alpha_{00}(r_i/r_0)^3}{3\{1 + [(r_i/r_0)^3 - 1]\alpha_{00}\}}\right\},$$

or substituting $r_i = r_0\{(1-\alpha_{00})\alpha_{0i}/[(1-\alpha_{0i})\alpha_{00}]\}^{1/3}$ from (4.8), we determine the bulk modulus

$$K_a = \left\{(1-\alpha_{0i})\chi_0 + \frac{3\alpha_{0i}}{3(P^0 - P_2) + (4\gamma/r_0)\,[\alpha_{00}(1-\alpha_{0i})/\alpha_{0i}(1-\alpha_{00})]^{1/3}}\right\}^{-1} \tag{4.9}$$

for a bubble suspension in the range $0 \leqslant \alpha_{0i} < \alpha_{0*}$. We find the condition under which this formula is applicable. Growth of bubbles in a volumetrically stretched liquid is a nonequilibrium process: The bubble radius and the pressure in the medium are not subject to condition (4.2), whereas $K_a$ characterizes, by definition, the elastic properties of a medium with slight deviations of its volume from the equilibrium value. Therefore, in determining $K_a$ ($0 \leqslant \alpha_{0i} \leqslant \alpha_{0*}$) using (4.9), one should consider this dependence for a spectrum of equilibrium states of the liquid medium, in each of which the bubble radii are related via (4.8) to $\alpha_{0i}$ and the pressure in the liquid is subject to condition (4.2).

A set of such model liquid samples containing bubbles with different values of $\alpha_{0i}$, which correspond to different stages of cavitation in a liquid medium being stretched, can be prepared under the condition of zero gravity [12], where the samples are able to retain their structure for at least 1 h (which is sufficient for experimental measurement of $K_a$). It is noteworthy that if the pressure in the liquid matrix of a sample is $P^0 > 10^4$ Pa, the bubble radii in a stably equilibrium state can take, for example, in the case of water, any values that satisfy the condition $r_i < 1.45$ cm, according to (4.2).

*The stage* $\alpha_{0*} \leqslant \alpha_0 < 1$. As we have already mentioned, when the concentration $\alpha_{0*}$ is attained, the medium's structure becomes a cellular skeleton in the process of subsequent volume stretching. However, since the volume of the liquid component remains incompressible upon deformation of the cellular skeleton formed, the elastic elements of the mechanical model $K_1$ and $K_2$ (Fig. 1b) are not deformed, and the model reduces to a two-component scheme that corresponds to a Voigt body: the elastic element $K_a$ and the piston $\zeta_1$, which are connected in parallel. It is natural that condition (4.2) is not applicable at this stage, and hence the dependence (4.9) is not applicable either.

For $\alpha_{0*} \leqslant \alpha_0 < 1$, the modulus $K_a$ must depend on the elasticity of the vapor-gas content (VGC) of the cells and also on the surface tension of the films separating these cells. We shall find the dependence of $K_a$ ($\alpha_{0*} \leqslant \alpha_0 < 1$) on the macrorheological characteristics and geometric parameters of the cells. It is known [13] that the moduli of bulk and shear elasticity of the medium can be determined via the Lamé coefficients

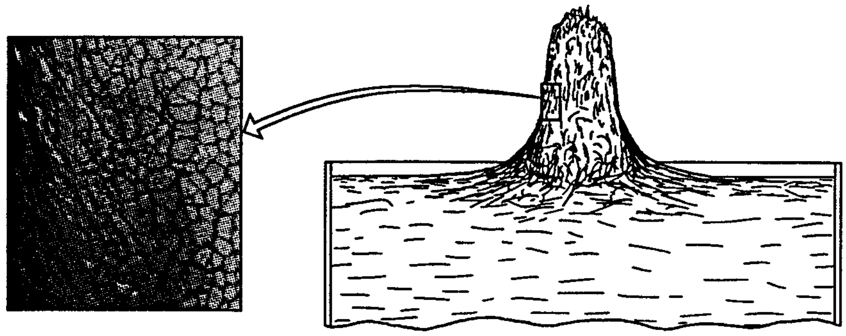

Fig. 3

by writing the relation for the free energy of the medium and differentiating this relation with respect to the strain tensor. In this way, Stomenovic and Wilson [14] derived a relation for the bulk modulus of a foam in the form

$$K = -V^0 \left(\frac{d\tilde{P}}{dV}\right)_0 - \frac{2}{9}\gamma \frac{S_1}{V^0}, \tag{4.10}$$

where $S_1/V^0$ is the ratio of the total surface of the cells to the foam volume in an unperturbed state, $\tilde{P}$ is the VGC pressure in the cells, and $V$ is the "perturbed" volume of the foam. Because the number of cells in the entire volume of the foam $N$ remains unchanged in the process of deviation from the equilibrium state, denoting the "perturbed" and "unperturbed" volumes of the cells by $V^+$ and $V^{0+}$, respectively, we have $V = NV^+$ and $V^0 = NV^{0+}$. With allowance for this, we obtain

$$-V^0 \left(\frac{d\tilde{P}}{dV}\right)_0 = -\frac{(d\tilde{P})_0}{(dV)_0/V^0} = -V^{0+} \left(\frac{d\tilde{P}}{dV^+}\right)_0. \tag{4.11}$$

On the other hand, by virtue of conservation of the VGC mass of a cell $V^+\tilde{\rho} = V^{0+}\tilde{\rho}^0$, where $\tilde{\rho}^0$ and $\tilde{\rho}$ are the "unperturbed" and "perturbed" densities of the VGC, we have $dV^+ = -V^{0+}\tilde{\rho}^0 d\tilde{\rho}/\tilde{\rho}^2$. Substituting this expression into (4.11), we obtain

$$-V^0 \left(\frac{d\tilde{P}}{dV}\right)_0 = -V^{0+} \left(\frac{\tilde{\rho}^2\, d\tilde{P}}{-V^{0+}\tilde{\rho}^0\, d\tilde{\rho}}\right)_0 = \tilde{\rho}^0 \left(\frac{d\tilde{P}}{d\tilde{\rho}}\right)_0 = \tilde{\rho}^0 \tilde{C}_0^2,$$

where $\tilde{C}_0$ is the velocity of sound in the VGC. With allowance for the last expression, Eq. (4.10) can be rewritten as

$$K = \tilde{\rho}^0 \tilde{C}_0^2 - \frac{2}{9}\gamma \frac{S_1}{V^0}. \tag{4.12}$$

To find the dependence of $S_1/V^0$ on $\alpha_0$ and the dimensions of the cells, it is necessary to determine the geometry of the cells. Figure 3 shows a photograph of the structure of a water sample that is being stretched in a pulsed regime in the zone of an unloading wave according to the scheme described in [15]. This is a typical case of formation of a cellular structure, which shows that by virtue of the nonequilibrium character of the process, the cells have different dimensions, and the structure of the medium is irregular. Since, by definition, $K_a$ characterizes the elasticity of the medium in the vicinity of a fixed equilibrium state, it seems expedient to choose a geometry of the cells that ensures minimum surface energy of the system, or values close to the minimum.

Bearing the aforesaid in mind, we assume that for $\alpha_0 \to \alpha_{0*}$, the bubbles remain spherical (which is quite justified if the bubbles grow slowly; such growth is generally accepted within the framework of construction of the bulk modulus of the medium) and form, similarly to droplets in a concentrated emulsion [16], a structure that corresponds to packing in which a bubble is inscribed in a rhomboidal dodecahedron (RD), touching each face at one point (Fig. 4a) (all the faces of the RD are identical rhombuses). The

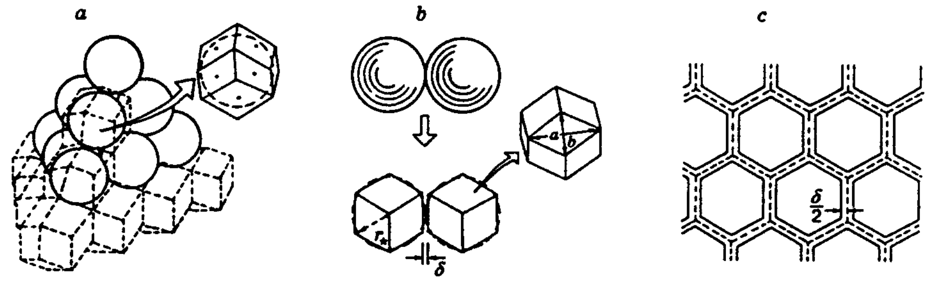


Fig. 4

dodecahedrons form a densely packed system, as is shown by the dotted curve in Fig. 4a. Since it is extremely complicated to describe exactly the process of continuous evolution of spherical bubbles to a cellular structure, the transition to the stage $\alpha_0 \geqslant \alpha_{0*}$ can be performed as follows.

We assume that when the packing stage $\alpha_{0*}$ is attained, every bubble immediately takes the shape of an RD of the same volume (Fig. 4b). Then, since the RD volume is defined by the relation [16]

$$V^+ = a^3/\sqrt{2}, \tag{4.13}$$

where $a$ is the principal diagonal of the rhombus, from $4\pi r_*^3/3 = a_*^3/\sqrt{2}$ we have

$$a_* = r_* \sqrt[3]{4\pi\sqrt{2}/3} \cong 1.8094 r_*. \tag{4.14}$$

Here and below, the asterisk denotes parameters that correspond to the stage $\alpha_0 = \alpha_{0*}$. If $N$ is the number of RD cells, their specific surface is determined by the relation

$$S_1^0/V^0 = S^+N/(V_0 + V^+N), \tag{4.15}$$

where, with allowance for the fact that the smaller diagonal of a rhomboidal face of an RD (Fig. 4b) is $b = a/\sqrt{2}$, the total surface of the RD faces is

$$S^+ = 12ab/2 = 3\sqrt{2}a^2. \tag{4.16}$$

At the stage $\alpha_0 \geqslant \alpha_{0*}$, all the liquid is concentrated in bridges between RD cells and, therefore, assuming them to be strictly regular in the medium's volume (Fig. 4c), we write

$$V_0 = S^+\delta N/2, \tag{4.17}$$

where $\delta$ is the thickness of the liquid bridges. From the definition of the volume concentration of the RD cells we have with allowance for (4.17)

$$\alpha_{0i} = \frac{V_i^+N}{V^0 + V_i^+N} = \left[1 + \frac{S_i^+\delta}{2V_i^+}\right]^{-1}.$$

Using (4.13) and (4.16), from the above relation we obtain

$$\delta_i = \frac{(1 - \alpha_{0i})}{3\alpha_{0i}} a_i. \tag{4.18}$$

With (4.13) and (4.16)–(4.18) taken into account, relation (4.15) takes the form

$$\frac{S_1^0}{V^0} = \frac{S_i^+N}{V_0 + V_i^+N} = 6\frac{\alpha_{0i}}{a_i}. \tag{4.19}$$

If the cells do not coalesce while expanding during the process of volume stretching (this is quite likely with allowance for the "splitting" pressure in the bridges), then from the condition $n_*V_*^0 = n_iV_i^0 = N = \text{const}$ and also $\alpha_{0i} = V_i^+N/V_i^0$, Eq. (4.13), and the expression for the moisture content in the medium

$\beta_i = 1 - \alpha_{0i} = V_0/V_i^0$, we have

$$n_i = n_* \frac{V_*^0}{V_i^0} = n_* \frac{V_0 + V_*^+ N}{V_0 + V_i^+ N} = \frac{n_*}{1 + (a_i^3/a_*^3 - 1)\alpha_{0*}}. \tag{4.20}$$

After that, using (4.13) and (4.20), we write

$$\alpha_{0i} = V_i^+ n_i = \frac{n_* a_i^3}{\sqrt{2}\,[1 + (a_i^3/a_*^3 - 1)\alpha_{0*}]}.$$

With allowance for $n_* = \alpha_{0*}/V_*^+ = \sqrt{2}\,\alpha_{0*}/a_*^3$, it follows from the above expression that

$$a_i = a_* \sqrt[3]{(1 - \alpha_{0*})\alpha_{0i}/(1 - \alpha_{0i})\alpha_{0*}}. \tag{4.21}$$

Replacing $a_i$ in (4.19) by $a_i$ from relation (4.21), with allowance for (4.14) and owing to the fact that the expressions $\alpha_{0*} = 4\pi r_*^3 N/[3(V_0 + 4\pi r_*^3 N/3)] = \alpha_{00} r_*^3/[r_0^3(1 + \alpha_{00} r_*^3/r_0^3)]$ give

$$r_* = r_0 \sqrt[3]{\alpha_{0*}/\alpha_{00}(1 - \alpha_{0*})},$$

we obtain the specific surface of the cells versus their concentration:

$$\frac{S_1^0}{V^0} = \frac{6}{r_* \sqrt[3]{(4\pi\sqrt{2})/3}} \sqrt[3]{\frac{1 - \alpha_{0i}}{1 - \alpha_{0*}} \alpha_{0*} \alpha_{0i}^2} = \frac{6}{r_0} \sqrt[3]{\frac{3}{4\sqrt{2}\pi} \alpha_{00}(1 - \alpha_{0i})\alpha_{0i}^2}. \tag{4.22}$$

Therefore, after (4.22) is substituted into (4.12), we obtain the expression

$$K_a(\alpha_{0*} \leqslant \alpha_{0i} < 1) \approx \tilde{\rho}^0 \tilde{C}_0^2 - \frac{2\gamma}{r_0} \sqrt[3]{\frac{\sqrt{2}}{9\pi} \alpha_{00}(1 - \alpha_{0i})\alpha_{0i}^2}. \tag{4.23}$$

Thus, from (4.9) and (4.23), we derive an expression that determines the dependence of the bulk modulus of a liquid containing bubbles or a foam structure provided that the condition of stable equilibrium for a two-phase medium is satisfied:

$$K_a = \begin{cases} \left[(1 - \alpha_{0i})\chi_0 + \dfrac{3\alpha_{0i}}{3(P^0 - P_2) + \dfrac{4\gamma}{r_0} \sqrt[3]{\dfrac{(1 - \alpha_{0i})\alpha_{00}}{(1 - \alpha_{00})\alpha_{0i}}}}\right]^{-1} & \text{for} \quad 0 \leqslant \alpha_{0i} < \alpha_{0*}, \\[4ex] \tilde{\rho}^0 \tilde{C}_0^2 - \dfrac{2\gamma}{r_0} \sqrt[3]{\dfrac{\sqrt{2}}{9\pi} \alpha_{00}(1 - \alpha_{0i})\alpha_{0i}^2} & \text{for} \quad \alpha_{0*} \leqslant \alpha_{0i} < 1. \end{cases} \tag{4.24}$$

Figure 5 shows dependences $K_a(\alpha_{0i})$ constructed by formula (4.24) for the case of a water matrix ($\gamma = 72.3$ g/sec$^2$ and $\chi_0^{-1} = K_0(\alpha_{0i} = 0) = 2.18 \cdot 10^9$ Pa). For the VGC we chose $\tilde{C}_0 = 4 \cdot 10^4$ cm/sec, and the density $\tilde{\rho}^0$ was determined taking into account the equilibrium state of the cells ($\tilde{P} = P^0$). Curves 1–3 refer to $P^0 = 10^5$, $10^4$, and $2 \cdot 10^3$ Pa, and the restriction imposed by the condition of stable equilibrium for monodisperse bubbles is $r_{0i} \leqslant 5 \cdot 10^{-3}$ cm, which, according to (4.8), corresponds to $\alpha_{0i} \leqslant 0.11$; curve 4 refers to $P^0 = 0$, and the relevant restriction is $\alpha_{0i} \leqslant 6 \cdot 10^{-6}$ ($r_{0i} \leqslant 4 \cdot 10^{-3}$ cm).

The segments $0 \leqslant \alpha_{0i} \leqslant \alpha_{0*}$ and $\alpha_{0*} \leqslant \alpha_{0i} < 1$ of curves 1 and 2 do not coincide exactly for $\alpha_{0i} = \alpha_{0*}$, because in deriving the dependence $K_a(\alpha_{0i})$, a jumplike transition from a bubble structure to a cellular structure for the medium is assumed. The points on curve 1 denote values calculated according to an existing formula for the bulk modulus [13]. As applied to the case considered, this dependence is of the form $K_a = \rho_0(1 - \alpha_{0i})C_0^2(\alpha_{0i})$, where $\rho_0$ is the density of the liquid; the velocity of sound in a bubble medium $C_0(\alpha_{0i})$ was determined based on experimental data of [17].

Thus, according to Fig. 5, the bulk modulus of a bubble suspension depends strongly on the pressure in the liquid matrix. If the pressure is assumed to be equal to zero (curve 4), the bubble medium being stretched

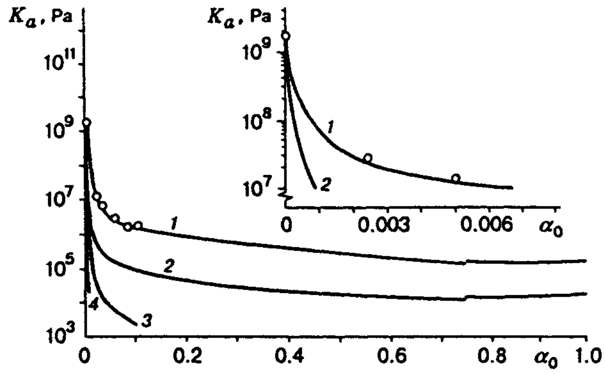


Fig. 5

can be considered a substance having no strength [9, 10]. However, in real processes, the atmospheric pressure, determining the bulk modulus of the medium, almost always acts on the expanding cavitating liquid volume. For example, the author and Chernobaev [18] have shown that if the value of the kinetic energy of a divergent flow of a cavitating liquid volume does not exceed a certain threshold value, the atmospheric counter pressure can lead to slowing down of the process of volume expansion of the medium and to collapse of bubble clusters.

A second rheological parameter that depends on $\alpha_0$ is the effective bulk viscosity $\zeta_1$. But this dependence can be found using experimental techniques.

**5.** We perform a qualitative analysis of the relaxation properties of a cavitating liquid, prescribing its instantaneous strain in the form

$$\varepsilon_V(t) = \varepsilon_0[U(t)], \tag{5.1}$$

where

$$[U(t)] = \begin{cases} 0 & \text{for} \quad t < t_1 = 0, \\ 1 & \text{for} \quad t \geqslant t_1; \end{cases} \qquad \frac{d[U(t)]}{dt} = [\delta(t)],$$

and $[\delta(t)]$ is the Dirac function. Substituting (5.1) into (3.4), with allowance for the properties of the $\delta$ function we obtain the equation

$$\ddot{\sigma}_V + \Big(\frac{1}{T_0} + \frac{K_\infty - K_a}{K_0 - K_a}\,\frac{1}{T_1}\Big)\dot{\sigma}_V + \frac{1}{T_0T_1}\sigma_V = f(t)$$

$$\Big[f(t) = \frac{K_a}{T_0T_1}\,\varepsilon_0\,[U(t)] + \Big(\frac{K_0}{T_0} + \frac{K_\infty - K_a}{K_0 - K_a}\,\frac{K_a}{T_1}\Big)\,\varepsilon_0\,[\delta(t)] + K_\infty\varepsilon_0\,\frac{d}{dt}\,[\delta(t)]\Big],$$

the general solution of which has the form

$$\sigma_V(t) = K_a\varepsilon_0[U(t)] + \frac{\varepsilon_0[U(t)]}{X_1 - X_2}\Big[\Big(\frac{K_a}{T_0T_1X_1} + \frac{K_0}{T_0} + \frac{K_\infty - K_a}{K_0 - K_a}\,\frac{K_a}{T_1} + K_\infty X_1\Big)\,\mathrm{e}^{X_1t}$$
$$- \Big(\frac{K_a}{T_0T_1X_2} + \frac{K_0}{T_1} + \frac{K_\infty - K_a}{K_0 - K_a}\,\frac{K_a}{T_1} + K_\infty X_2\Big)\,\mathrm{e}^{X_2t}\Big]. \tag{5.2}$$

Here

$$X_1 = -\frac{1}{2}\Big(\frac{1}{T_0} + \frac{1}{T_1}\,\frac{K_\infty - K_a}{K_0 - K_a} - D\Big), \quad X_2 = -\frac{1}{2}\Big(\frac{1}{T_0} + \frac{1}{T_1}\,\frac{K_\infty - K_a}{K_0 - K_a} + D\Big),$$
$$D = \sqrt{\frac{1}{T_0^2} + \frac{1}{T_1^2}\Big(\frac{K_\infty - K_a}{K_0 - K_a}\Big)^2 + \frac{2}{T_0T_1}\,\frac{K_\infty + K_a - 2K_0}{K_0 - K_a}}. \tag{5.3}$$

According to (5.2) and (5.3), we have

$$\sigma_V(t) \to \begin{cases} K_\infty \varepsilon_0 & \text{for} \quad t \to +0, \\ K_a \varepsilon_0 & \text{for} \quad t \to \infty, \end{cases}$$

i.e., the tensile stress decreases with time to a constant value: a relaxation process occurs in the medium, which corresponds to the mechanical model of the medium (see Fig. 2).

The relaxation time of tensile stresses $T$ is found from the condition

$$\sigma_V(t = T) = \sigma_V(t = 0)/e \qquad (\mathrm{e} = 2.71828). \tag{5.4}$$

Substituting (5.4) into (5.2), we have

$$\begin{aligned} &\Big(\frac{K_a}{T_0 T_1 X_2} + \frac{K_0}{T_0} + \frac{K_\infty - K_a}{K_0 - K_a}\frac{K_a}{T_1} + K_\infty X_1\Big)\mathrm{e}^{X_1 T} \\ &- \Big(\frac{K_a}{T_0 T_1 X_2} + \frac{K_0}{T_0} + \frac{K_\infty - K_a}{K_0 - K_a}\frac{K_a}{T_1} + K_\infty X_2\Big)\mathrm{e}^{X_2 T} = \Big(\frac{K_\infty}{\mathrm{e}} - K_a\Big)(X_1 - X_2), \end{aligned} \tag{5.5}$$

where $X_1$ and $X_2$ are calculated from (5.3) with allowance for $T_0 = \zeta_0/(K_\infty - K_0)$ and $T_1 = \zeta_1/(K_\infty - K_a)$. To determine $T$ in this way, we have to know numerical values of the rheological parameters of the medium that enter (5.4) and (5.5). Here $K_\infty$, $K_0$, and $\zeta_0$ are, as a rule, known for most liquids, and $K_a(\alpha_0)$ can be calculated by formula (4.24). But the determination of $\zeta_0$ from (1.1) in the entire range of $\alpha_0$ seems to be incorrect, because for $\alpha_0 = \alpha_{00} \to 0$, Eq. (1.1) has a singularity in the approximation of an incompressible matrix, the shape of the bubbles deviates from spherical for large initial values $\alpha_0 = \alpha_{00}$, and their interaction should be taken into account, which is not envisaged in deriving formula (1.1). We note that for $\alpha_{00} = 10^{-4}$ and $r_0 = 10^{-5}$ cm, the value of $T$ calculated from (5.5) with allowance for (1.1), (4.24), and (5.3) is equal to $1.8 \cdot 10^{-8}$ sec, which is very close to the result reported in [8], where $T = 0.63 \cdot 10^{-8}$ sec.

In view of this, the next step in work on the development of physical and mathematical models of a cavitating liquid medium being stretched is the creation of rheological methods that would allow one to find empirical dependences of the rheological parameters of the medium, including $\zeta_1$, on $\alpha_0$ and the strain rate of the medium.

This work was supported by the Russian Foundation for Fundamental Research (Grant No. 96-01-01772).

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